De Morgan Laws for Sets

De Morgan Laws for sets are stated as,

(1) (A ∪ B)' = A' ∩ B'

(2) (A ∩ B)' = A' ∪ B'

We will seek the validity of these theorems by investigating case by case of the relative compositions of the acting sets A and B,

The sets A and B may be either equal , disjoint or one is a proper subset of the other. No other configuration is possible.

We will always use the union of the sets A and B for the universe of discourse.

1 - First De Morgan Law

Case 1 : A = B

In this case, (A ∪ B) = A = B , (A ∪ B)' = ∅ , A' = B' = ∅ , A' ∩ B' = ∅ ∩ ∅ = ∅

Finally,

∅ = ∅ Q.E.D.

Case 2 : A and B are Disjoint

In this case, (A ∪ B) = U , (A ∪ B)' = U' = ∅ , A' = B , B' = A , A' ∩ B' = A ∩ B = ∅

Finally,

∅ = ∅ Q.E.D.

Case 2 : A ⊂ B (A is a Proper Subset of B)

In this case, (A ∪ B) = U = B , (A ∪ B)' = U' = ∅ , A' = (B - A) , B' = ∅ , A' ∩ B' = ((B-A) ∩ ∅) = ∅

Finally,

∅ = ∅ Q.E.D.

General Result for The First De Morgan Law

First De Morgan Law for Sets,

(A ∪ B)’ = A’ ∩ B’

Is a valid theorem for all compositions of both A and B.

2 - Second De Morgan Law

Second De Morgan Law is stated as,

(A ∩ B)’ = A’ ∪ B’

When A ∪ B is taken as the universe of discourse in an arbitrary length, the intersection of A and B results with a set which the composition is dependent of the relative compositions of the sets under consideration.

We will check the logical validiy of this rule, case by case, until all the possible relative configurations of the sets under consideration are all depleted.

Case 1 : A = B

In this case, A ∩ B = A = B

(A ∩ B)’ = ∅ ,

A’ = ∅ , B’ = ∅ , A’ ∪ B ‘ = ∅

Therefore,

(A ∩ B)’ = A’ ∪ B’ = ∅   Q.E.D.

Case 2 : A and B are Disjoint Sets

In this case, (A ∩ B) = ∅

(A ∩ B)’ = ∅’ = U (Universal Set)

A' = B , B’ = A , (A’ ∪ B’) = (A ∪ B) = U

Therefore,

(A ∩ B)’ = A’ ∪ B’ = U   Q.E.D.

Case 3 : A is a Proper Set of B ($A \subset B$)

In this case, $(A \cap B) = A$, $(A \cap B)' = (B - A)$

$A' = (B - A)$, $B' = \emptyset$, $A' \cup B' = (B - A)$

Therefore,

$(A \cap B)' = A' \cup B' = (B-A)$ Q.E.D.

General Result for The Second De Morgan Law for Sets

Second De Morgan Law for Sets,

$(A \cap B)' = A' \cup B'$

Is a valid theorem for all compositions of both A and B.

General Result for The De Morgan Laws for Sets

De Morgan Laws for Sets,

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

Are valid theorems independent of compositions of both A and B.

Discleamer : I hereby certify that no other proofs is consulted before setting up this proof and this is purely an original work. But it is possible that similar proofs may be given elsewhere without the knowledge of the author.

Numerical Evaluations by the Application of the Mathematica

Mathematica (11.3) is used to investigate numerical evaluations of both De Morgan Laws for sets. For the mathematical point of view the logical proofs are essential and numerical correspondences are futile. But numerical examples are useful for better understanding mathematical results of the methods applied in the logical proofs. These numerical results are only given here under these auspices.

First De Morgan Law

Case 1 : A = B

```
In[316]:= ClearAll[A, B]

In[317]:= A = {1, 2, 3}; B = {1, 2, 3};

In[318]:= Union[A , B]
Out[318]= {1, 2, 3}

In[319]:= Complement[A⋃ B , A ⋃ B]
Out[319]= { }

In[320]:= Complement[A⋃ B , A]
Out[320]= { }

In[321]:= Complement[A⋃ B , B]
Out[321]= { }

In[322]:= (Complement[ A⋃ B, A]) ⋂ (Complement[A⋃ B, B])
Out[322]= { }

In[323]:= Complement[A⋃ B , A ⋃ B] ==
           (Complement[ A⋃ B, A ]) ⋂ (Complement[A⋃ B, B])
Out[323]= True
```

Case 2 : A and B are Disjoint

```
In[324]:= ClearAll[A, B]

In[325]:= A = {1, 2, 3}; B = {4, 5, 6};

In[326]:= Union[A , B]
Out[326]= {1, 2, 3, 4, 5, 6}

In[327]:= Complement[A⋃ B , A ⋃ B]
Out[327]= { }
```

```
In[328]:= Complement[A ⋃ B , A]
Out[328]= {4, 5, 6}

In[329]:= Complement[A ⋃ B , B]
Out[329]= {1, 2, 3}

In[330]:= (Complement[ A ⋃ B, A]) ⋂ (Complement[A ⋃ B, B])
Out[330]= { }

In[331]:= Complement[A ⋃ B , A ⋃ B] ==
           (Complement[ A ⋃ B, A ]) ⋂ (Complement[A ⋃ B, B])
Out[331]= True
```

Case 3 : A ⊂ B

```
In[332]:= ClearAll[A, B]

In[333]:= A = {1, 2, 3}; B = {1, 2, 3, 4, 5, 6};

In[334]:= Union[A , B]
Out[334]= {1, 2, 3, 4, 5, 6}

In[335]:= Complement[A ⋃ B , A ⋃ B]
Out[335]= { }

In[336]:= Complement[A ⋃ B , A]
Out[336]= {4, 5, 6}

In[337]:= Complement[A ⋃ B , B]
Out[337]= { }

In[338]:= (Complement[ A ⋃ B, A]) ⋂ (Complement[A ⋃ B, B])
Out[338]= { }

In[339]:= Complement[A ⋃ B , (A ⋃ B)] ==
           Complement [A ⋃ B, A] ⋂ Complement [A ⋃ B, B]
Out[339]= True
```

Second De Morgan Law

Case 1 : A = B

```
In[340]:= ClearAll[A, B]

In[341]:= A = {1, 2, 3}; B = {1, 2, 3};

In[342]:= Intersection[A , B]

Out[342]= {1, 2, 3}

In[343]:= Complement[A⋃ B , A ⋂ B]

Out[343]= { }

In[344]:= Complement[A⋃ B , A]

Out[344]= { }

In[345]:= Complement[A⋃ B , B]

Out[345]= { }

In[346]:= (Complement[ A⋃ B, A]) ⋃ (Complement[A⋃ B, B])

Out[346]= { }

In[347]:= Complement[A⋃ B , A ⋂ B] ==
            (Complement[ A⋃ B, A ]) ⋃ (Complement[A⋃ B, B])

Out[347]= True
```

Case 2 : A and B are Disjoint Sets

```
In[348]:= ClearAll[A, B]

In[349]:= A = {1 , 2 , 3}; B = {4 , 5 , 6};

In[350]:= Intersection[A , B]

Out[350]= { }

In[351]:= Complement[A⋃ B , A ⋂ B]

Out[351]= {1, 2, 3, 4, 5, 6}

In[352]:= Complement[A⋃ B , A]

Out[352]= {4, 5, 6}
```

```
In[353]:= Complement[A⋃ B , B]
Out[353]= {1, 2, 3}

In[354]:= (Complement[ A⋃ B, A]) ⋃ (Complement[A⋃ B, B])
Out[354]= {1, 2, 3, 4, 5, 6}

In[355]:= Complement[A⋃ B , A ⋂ B] ==
            (Complement[ A⋃ B, A ]) ⋃ (Complement[A⋃ B, B])
Out[355]= True
```

Case 3 : A is a Proper Set of B (A ⊂ B)

```
In[356]:= ClearAll[A, B]

In[357]:= A = {1 , 2 , 3}; B = {1 , 2 , 3 , 4 , 5 , 6};

In[358]:= Union[A , B]
Out[358]= {1, 2, 3, 4, 5, 6}

In[359]:= Intersection[A , B]
Out[359]= {1, 2, 3}

In[360]:= Complement[A⋃ B , A ⋂ B]
Out[360]= {4, 5, 6}

In[361]:= Complement[A⋃ B , A]
Out[361]= {4, 5, 6}

In[362]:= Complement[A⋃ B , B]
Out[362]= { }

In[363]:= (Complement[ A⋃ B, A]) ⋃ (Complement[A⋃ B, B])
Out[363]= {4, 5, 6}

In[364]:= Complement[A⋃ B , A ⋂ B] ==
            (Complement[ A⋃ B, A ]) ⋃ (Complement[A⋃ B, B])
Out[364]= True
```